

ALGEBRAIC PROPERTIES OF UNIVERSAL SQUAREFREE LEXSEGMENT IDEALS ¹

MARILENA CRUPI AND MONICA LA BARBIERA

ABSTRACT. Let K be a field and let $A = K[X_1, \dots, X_n]$ be the polynomial ring in X_1, \dots, X_n with coefficients in the field K . We study the universal squarefree lexsegment ideals. We put our attention on their combinatorics computing some invariants. Moreover we study the link between such special class of squarefree lexsegment ideals and the so called s -sequences.

INTRODUCTION

Let K be a field and let $A = K[X_1, \dots, X_n]$ be the polynomial ring in X_1, \dots, X_n with coefficients in the field K . Set $A_{[m]} = K[X_1, \dots, X_n, X_{n+1}, \dots, X_{n+m}]$, where m is a positive integer. A squarefree lexsegment ideal I of A is called *universal squarefree lexsegment ideal* (abbreviated USLI), if for any integer $m \geq 1$, the squarefree monomial ideal $IA_{[m]}$ of the polynomial ring $A_{[m]}$ is a squarefree lexsegment ideal. Let M^s denote the set of all squarefree monomials in the variables X_1, \dots, X_n . A squarefree lexsegment ideal I of A with $G(I) = \{u_1, \dots, u_\ell, u_{\ell+1}\}$, $u_1 > \dots > u_\ell > u_{\ell+1}$ with respect to the homogeneous lexicographic order on M^s , is called *almost universal squarefree lexsegment ideal* (abbreviated AUSLI), if I is not an USLI of A but the ideal $J = (u_1, \dots, u_\ell)$ is an USLI of A . $G(I)$ is the unique minimal set of monomial generators of the monomial ideal I . These definitions were introduced by Babson, Novik and Thomas in [2] in order to study the symmetric version of algebraic shifting. The algebraic shifting is an algebraic operation introduced by Kalai ([4], [13]) that transforms a simplicial complex into a simpler complex that preserves important combinatorial, topological and algebraic invariants.

In this paper we put our attention on the structure of universal squarefree lexsegment ideals (Characterization 2.1). We analyze their combinatorics in order to compute some invariants as the projective dimension, the Castelnuovo-Mumford regularity and the depth (Corollary 2.6).

In [11], the authors introduced the concept of s -sequences in order to study the symmetric algebra of a module M on a noetherian ring R . One of their motivation was that is a difficult problem to compute standard algebraic invariants of the graded algebra $\text{Sym}_R(M)$. Their proposal was to determine these invariants in terms of the corresponding invariants of special quotients of the ring R . The s -sequences are an important tool for this computation.

2000 *Mathematics Subject Classification.* 13A02, 13B25, 13C15, 13D08.

Key words and phrases. Monomial ideals, squarefree lexicographic ideals, minimal resolutions, s -sequences, standard invariants.

¹ To appear in *Algebra Colloquium*.

In this paper we analyze the problem when a squarefree lexsegment ideal I of degree d of the polynomial ring A is generated by an s -sequence. We are able to state that this happens if I is an USLI or an AUSLI (Theorem 3.4). Consequently their symmetric algebra is studied (Theorems 3.9 and 3.11).

The structure of the paper is organized as follows.

In section 1, we recall some notions that we will use during the paper.

In section 2, we describe in a suitable way the USLIs (Characterization 2.1). Hence we state a characterization of an USLI of degree d (Proposition 2.3). Moreover, we analyze some invariants associated to the universal squarefree lexsegment ideals. The main result states that an USLI $I \subsetneq A$ has a unique extremal Betti number whose value is 1 (Proposition 2.5). This fact allows us to compute $\text{proj}_A(I)$, $\text{reg}_A(I)$ and $\text{depth}_A(A/I)$ (Corollary 2.6).

Section 3 is dedicated to the symmetric algebra of USLIs and AUSLs of the polynomial ring A . More precisely let I be a lexsegment ideal of A generated by squarefree monomials in a same degree, we establish that the ideal I is generated by an s -sequence if and only if I is an USLI or an AUSLI (Theorem 3.4). This result is proved using the characterization of the monomial s -sequences by the Gröbner bases. As a consequence of this result we study the problem of computing standard algebraic invariants of the graded algebra $\text{Sym}_A(I)$ when I is a squarefree lexsegment ideal generated by an s -sequence. More precisely, we give a formula for the dimension and the multiplicity of $\text{Sym}_A(I)$ when I is an AUSLI (Theorem 3.11). A formula for the Castelnuovo-Mumford regularity and for the depth of $\text{Sym}_A(I)$ when I is an USLI (Theorem 3.9) is also stated.

1. PRELIMINARIES AND NOTATIONS

Let K be a field and let $A = K[X_1, \dots, X_n]$ be the polynomial ring in X_1, \dots, X_n with coefficients in the field K . We consider A as an \mathbb{N} -graded ring and each $\deg X_i = 1$. We denote by M_d the set of all monomials of degree d of the polynomial ring A . If $I \subsetneq A$ is a monomial ideal we denote by $G(I)$ the unique minimal set of monomial generators of I and by $G(I)_d$ the set $G(I)_d = \{u \in G(I) : \deg u = d\}$, for $d > 0$.

For a monomial $1 \neq u \in A$, we set

$$\text{supp}(u) = \{i : X_i \text{ divides } u\}.$$

$$m(u) = \max\{i : X_i \text{ divides } u\}.$$

Recall that a squarefree monomial ideal I of A is called *squarefree stable* if for all $u \in G(I)$, one has $(X_j u)/X_{m(u)} \in I$ for all $j < m(u)$ with $j \notin \text{supp}(u)$ ([1]).

Now let M_d^s denote the set of all squarefree monomials of degree $d \geq 1$ in the variables X_1, \dots, X_n . We write $>_{\text{slex}}$ for the *lexicographic order* on the finite set M_d^s , that is, if $u = X_{i_1} \cdots X_{i_d}$ and $v = X_{j_1} \cdots X_{j_d}$ are squarefree monomials belonging to M_d^s with $1 \leq i_1 < i_2 < \cdots < i_d \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_d \leq n$, then $u >_{\text{slex}} v$ if $i_1 = j_1, \dots, i_{s-1} = j_{s-1}$ and $i_s < j_s$ for some $1 \leq s \leq d$ ([1]).

Let M^s be the set of all squarefree monomials in the variables X_1, \dots, X_n . We denote by $>_{\text{hslex}}$ the *homogeneous lexicographic order* on M^s , that is, if $u, v \in M^s$, then $u >_{\text{hslex}} v$ if $\deg u > \deg v$, or if $\deg u = \deg v$ and $u >_{\text{slex}} v$.

A monomial ideal $I \subsetneq A$ is called a *squarefree lexsegment ideal* if I is generated by squarefree monomials, and for all squarefree monomials $u \in I$ and all squarefree

monomials $v \in A$ with $\deg u = \deg v$ and $v >_{\text{slex}} u$, then $v \in I$. Every squarefree lexsegment ideal of A is obviously a squarefree stable ideal.

Set $A_{[m]} = K[X_1, \dots, X_n, X_{n+1}, \dots, X_{n+m}]$, where m is a positive integer.

We quote the next definitions from [2].

Definition 1.1. A squarefree lexsegment ideal I of A is called *universal squarefree lexsegment ideal* (USLI, for short), if for any integer $m \geq 1$, the squarefree monomial ideal $IA_{[m]}$ of the polynomial ring $A_{[m]}$ is a squarefree lexsegment ideal.

In other words a universal squarefree lexsegment ideal of A is a squarefree lexsegment ideal I of A which remains being squarefree lexsegment if we regard I as an ideal of the polynomial ring $A_{[m]}$ for all $m \geq 1$.

Example 1.2. (1) The squarefree lexsegment ideal $I = (X_1X_2, X_1X_3X_4)$ of $A = K[X_1, X_2, X_3, X_4]$ is an USLI. Indeed I is a squarefree lexsegment ideal of the polynomial ring $A_{[m]}$ for all $m \geq 1$.

(2) The squarefree lexsegment ideal $I = (X_1X_2, X_1X_3X_4, X_2X_3X_4)$ of $A = K[X_1, X_2, X_3, X_4]$ is not an USLI. Indeed I is not a squarefree lexsegment ideal of the polynomial ring $A_{[1]} = K[X_1, X_2, X_3, X_4, X_5]$. In fact $X_1X_4X_5 >_{\text{slex}} X_2X_3X_4$ and $X_1X_4X_5 \notin IA_{[1]}$.

Definition 1.3. A squarefree lexsegment ideal I of A with $G(I) = \{u_1, \dots, u_\ell, u_{\ell+1}\}$, $u_1 >_{\text{hslex}} \dots >_{\text{hslex}} u_\ell >_{\text{hslex}} u_{\ell+1}$, is called *almost universal squarefree lexsegment ideal* (AUSLI, for short), if I is not an USLI of A but the ideal $J = (u_1, \dots, u_\ell)$ is an USLI of A .

Example 1.4. The squarefree lexsegment ideal $I = (X_1X_2, X_1X_3X_4, X_2X_3X_4)$ of $A = k[X_1, X_2, X_3, X_4]$ is an AUSLI. Indeed I is not an USLI of A , but the ideal $J = (X_1X_2, X_1X_3X_4)$ is an USLI of A (Example 1.2, (1)).

We finish this section recall the notion of extremal Betti numbers of a graded ideal I of the polynomial ring A .

If I is a graded ideal of A , then I has a minimal graded free A -resolution

$$F : 0 \rightarrow F_s \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow I \rightarrow 0$$

where $F_i = \bigoplus_{j \in \mathbb{Z}} A(-j)^{\beta_{i,j}}$.

The integers $\beta_{i,j} = \beta_{i,j}(I) = \dim_K \text{Tor}_i(K, I)_j$ are called the graded Betti numbers of I , while $\beta_i(I) = \sum_j \beta_{i,j}(I)$ are called the total Betti numbers of I .

To a graded ideal I two invariants can be associated the *projective dimension* and the *Castelnuovo-Mumford regularity* ([5], [9]) that are defined, respectively, as follows:

$$\text{proj}_A(I) = \max\{i : \beta_i(I) \neq 0\},$$

$$\text{reg}_A(I) = \max\{j - i : \beta_{i,j}(I) \neq 0\} = \max\{j : \beta_{i,i+j}(I) \neq 0, \text{ for some } i \in \mathbb{N}\}.$$

Bayer, Charalambous and Popescu introduced in [3] a refinement of the invariants above defined, giving the notion of extremal Betti numbers.

Definition 1.5. A Betti number $\beta_{k,k+\ell}(I) \neq 0$ is called *extremal* if $\beta_{i,i+j}(I) = 0$ for all $i \geq k$, $j \geq \ell$, $(i, j) \neq (k, \ell)$.

If $\beta_{k_1, k_1 + \ell_1}(I), \dots, \beta_{k_t, k_t + \ell_t}(I)$, $k_1 > \dots > k_t, \ell_1 < \dots < \ell_t$, are all extremal Betti numbers of I , then $k_1 = \text{proj}_A(I)$ and $\ell_t = \text{reg}_A(I)$.

The following characterization of the extremal Betti numbers of squarefree stable ideals was given in [8, Proposition 4.1].

Proposition 1.6. *Let $I \subsetneq A$ be a squarefree stable ideal. The following conditions are equivalent:*

- (1) $\beta_{k, k+\ell}(I)$ is extremal.
- (2) $k + \ell = \max\{m(u) : u \in G(I)_\ell\}$ and $m(u) < k + j$ for all $j > \ell$ and $u \in G(I)_j$.

As a consequence of the above result, we obtain the following.

Corollary 1.7. *Let $I \subsetneq A$ be a squarefree stable ideal.*

- (1) *If $\beta_{k, k+\ell}(I)$ is an extremal Betti number of I , then*

$$\beta_{k, k+\ell}(I) = |\{u \in G(I)_\ell : m(u) = k + \ell\}|.$$

- (2) *Set $d = \max\{j : G(I)_j \neq \emptyset\}$ and $m = \max\{m(u) : u \in G(I)\}$, then $\beta_{m-d, m-d+d}(I)$ is the unique extremal Betti number of I if and only if $m = \max\{m(u) : u \in G(I)_d\}$ and for every $w \in G(I)_j$, $j < d$, $m(w) < m$.*

2. UNIVERSAL SQUAREFREE LEXSEGMENT IDEALS

In this section we discuss the combinatorics of universal squarefree lexsegment ideals. Moreover we compute some standard invariants.

In [2, Definition 4.1] there is a characterization of USLIs. In order to reformulate it for our purpose, we need to introduce some notations.

For a sequence of non negative integers $(k_i)_{i \in \mathbb{N}}$, we define the following set:

$$\text{supp}(k_i)_{i \in \mathbb{N}} = \{i \in \mathbb{N} : k_i \neq 0\}.$$

If $\text{supp}(k_i)_{i \in \mathbb{N}} = \{d_1, \dots, d_t\}$, with $d_1 < d_2 < \dots < d_t$, then we associate to $(k_i)_{i \in \mathbb{N}}$ the following integers:

$$R_j = j + \sum_{i=1}^j k_i$$

for $j = 1, \dots, d_t$. We set $R_j = 0$, for $j > d_t$.

Hence we can reformulate the characterization contained in [2, Definition 4.1], as follows:

Characterization 2.1. *Let $I \subsetneq A$ be an ideal generated in degrees $d_1 < d_2 < \dots < d_t$. Then I is an USLI of A if and only if*

$$G(I)_{d_i} = \left\{ \left(\prod_{j=1}^{d_i-1} X_{R_j} \right) X_\ell : \ell \in [R_{d_{i-1}} + 1, R_{d_i} - 1] \right\}, \text{ for } i = 1, \dots, t,$$

where $R_j = j + \sum_{i=1}^j |G(I)_{d_i}|$, for $j = 1, \dots, d_t$.

The characterization above follows from the statement contained in [2, Definition 4.1], choosing $(k_i)_{i \in \mathbb{N}}$ as the sequence of non negative integers such that $\text{supp}(k_i)_{i \in \mathbb{N}} = \{d_1, \dots, d_t\}$ and $k_{d_i} = |G(I)_{d_i}|$, for $i = 1, \dots, t$.

Remark 2.2. Assume that $(k_i)_{i \in \mathbb{N}}$ is a sequence of non negative integers such that

$$\text{supp}(k_i)_{i \in \mathbb{N}} = \{d_1, \dots, d_t\}, \quad d_1 < d_2 < \dots < d_t.$$

Then there exists an USLI $I \subsetneq A = K[X_1, \dots, X_n]$ generated in degrees d_1, \dots, d_t such that $|G(I)_{d_i}| = k_{d_i}$, for $i = 1, \dots, t$ if and only if $n \geq d_t + \sum_{i=1}^{d_t} k_i - 1$.

Thanks to the above statements we can give the following characterizations of an USLI generated in a same degree d .

Proposition 2.3. *Let I be a squarefree lexsegment ideal of $A = K[X_1, \dots, X_n]$ generated in degree d . Then I is an USLI of A if and only if $|G(I)| \leq n - d + 1$.*

Proof. Let $(k_i)_{i \in \mathbb{N}}$ be the sequence of non negative integers such that $\text{supp}(k_i)_{i \in \mathbb{N}} = \{d\}$, with $k_d = |G(I)_d| = |G(I)|$. From Remark 2.2, $I \subsetneq A$ is an USLI generated in degree d if and only if $n \geq d + k_d - 1 = d + |G(I)| - 1$ that is if and only if $|G(I)| \leq n - d + 1$. □

Note that if $I \subsetneq A = K[X_1, \dots, X_n]$ is an USLI generated in degree d then

$$(2.1) \quad G(I) = \{X_1 X_2 \cdots X_{d-1} X_d, X_1 X_2 \cdots X_{d-1} X_{d+1}, \dots, X_1 X_2 \cdots X_{d-1} X_k\},$$

with $d \leq k \leq n$.

Moreover if I is an AUSLI generated in degree d , then

$$(2.2) \quad G(I) = \{X_1 X_2 \cdots X_{d-1} X_d, \dots, X_1 X_2 \cdots X_{d-1} X_n, X_1 X_2 \cdots X_{d-2} X_d X_{d+1}\}.$$

Remark 2.4. It is clear that a squarefree lexsegment ideal $I \subsetneq A$ generated in degree d is an AUSLI if and only if $|G(I)| = n - d + 2$.

We finish this section computing some invariants of an USLI I by its extremal Betti numbers. In general squarefree lexsegment ideals may have more than just one extremal Betti number ([7],[8]).

Consider, for example the squarefree ideal

$$I = (X_1 X_2, X_1 X_3, X_1 X_4, X_1 X_5, X_1 X_6, X_1 X_7, X_2 X_3 X_4, X_2 X_3 X_5, X_2 X_3 X_6, X_2 X_3 X_7, \\ X_2 X_4 X_5 X_6, X_2 X_4 X_5 X_7, X_3 X_4 X_5 X_6 X_7)$$

of $K[X_1, \dots, X_7]$. It is a squarefree lexsegment ideal with $\beta_{5,5+2} = 1$, $\beta_{4,4+3} = 1$, $\beta_{3,3+4} = 1$, $\beta_{2,2+5} = 1$ as extremal Betti numbers.

For an USLI, we can state.

Proposition 2.5. *Let $I \subsetneq A$ be an USLI. Then I has an unique extremal Betti number whose value is equal to 1.*

Proof. Let I be an USLI generated in degrees $d_1 < d_2 < \dots < d_t$. From Theorem 2.1, $\ell = R_{d_t} - 1 = \max\{m(u) : u \in G(I)_{d_t}\}$, where $d_t = \max\{j : G(I)_j \neq \emptyset\}$ and $\ell = \max\{m(u) : u \in G(I)\}$. Hence, from Corollary 1.7, $\beta_{\ell-d_t, \ell-d_t+d_t}(I)$ is its unique extremal Betti number and its value is 1. □

Corollary 2.6. *Let $I \subsetneq A$ be an USLI generated in degrees $d_1 < d_2 < \dots < d_t$. Then*

- (1) $\text{proj}_A(I) = |G(I)| - 1$ and $\text{reg}_A(I) = d_t$.
- (2) $\text{depth}_A(A/I) = n - |G(I)|$.

Proof. (1). From Proposition 2.5, $\beta_{\ell-d_t, \ell-d_t+d_t}(I)$, with $\ell = \max\{m(u) : u \in G(I)_{d_t}\}$, is the unique extremal Betti number of I . Hence $\ell - d_t = \text{proj}_A(I)$ and $d_t = \text{reg}_A(I)$. With the same notations of Characterization 2.1, we have that

$$\text{proj}_A(I) = \ell - d_t = d_t + \sum_{i=1}^t |G(I)_{d_i}| - d_t - 1 = |G(I)| - 1.$$

(2). It follows from the Auslander-Buchsbaum formula. \square

Remark 2.7. Recall that for a squarefree stable ideal $I \subsetneq A$, $\text{reg}_A(I) = \max\{\deg u : u \in G(I)\}$ [1, Corollary 2.6].

3. USLIs, AUSLIs AND s -SEQUENCES

In this section we study the strict link between USLIs (resp. ASLIs) and s -sequences. We compute standard algebraic invariants of the graded algebra $\text{Sym}_A(I)$ when I is an USLI or an AUSLI of degree d in terms of the annihilator ideals of the s -sequence that generates I .

Let A be a noetherian ring, M be a finitely generated A -module with generators f_1, \dots, f_q . For every $i = 1, \dots, q$, we set $M_{i-1} = Af_1 + \dots + Af_{i-1}$ and let $I_i = M_{i-1} :_A f_i$ be the colon ideal. We set $I_0 = (0)$. Since $M_i/M_{i-1} \simeq A/I_i$, so I_i is the annihilator of the cyclic module A/I_i . I_i is called an annihilator ideal of the sequence f_1, \dots, f_q .

Let $\text{Sym}_A(M)$ be the symmetric algebra of M . Let (a_{ij}) , for $i = 1, \dots, q$, $j = 1, \dots, p$, be the relation matrix of M . It is known that the symmetric algebra $\text{Sym}_A(M)$ has a presentation $A[T_1, \dots, T_q]/J$, with $J = (g_1, \dots, g_p)$ where $g_j = \sum_{i=1}^q a_{ij}T_i$ for $j = 1, \dots, p$.

We consider $S = A[T_1, \dots, T_q]$ a graded ring by assigning to each variable T_i degree 1 and to the elements of A degree 0. Then J is a graded ideal and the natural epimorphism $S \rightarrow \text{Sym}_A(M)$ is a homomorphism of graded A -algebras.

Let $<$ be a monomial order on the monomials in the variables T_i such that $T_1 < T_2 < \dots < T_q$. With respect to this term order, for any polynomial $f = \sum a_\alpha \underline{T}^\alpha \in S$, where $\underline{T}^\alpha = T_1^{\alpha_1} \dots T_q^{\alpha_q}$ and $\alpha = (\alpha_1, \dots, \alpha_q) \in \mathbb{N}^q$, we put $\text{in}_<(f) = a_\alpha \underline{T}^\alpha$, where \underline{T}^α is the largest monomial in f such that $a_\alpha \neq 0$.

So we can define the monomial ideal $\text{in}_<(J) = (\text{in}_<(f) | f \in J)$. Notice that $(I_1T_1, I_2T_2, \dots, I_qT_q) \subseteq \text{in}_<(J)$ and the two ideals coincide in degree 1.

Definition 3.1. The generators f_1, \dots, f_q of M are called an s -sequence (with respect to an admissible term order $<$) if

$$(I_1T_1, I_2T_2, \dots, I_qT_q) = \text{in}_<(J).$$

If $I_1 \subseteq I_2 \subseteq \dots \subseteq I_q$, the sequence is a strong s -sequence.

Now, let $A = K[X_1, \dots, X_n]$ be the polynomial ring over a field K and let $<$ any term order on $K[X_1, \dots, X_n; T_1, \dots, T_q]$ with $X_1 > \dots > X_n$, $T_1 < T_2 < \dots < T_q$, $X_i < T_j$ for all i and j . Then for any Gröbner basis G of $J \subsetneq K[X_1, \dots, X_n; T_1, \dots, T_q]$ with respect to $<$, we have $\text{in}_<(J) = (\text{in}_<(f) | f \in G)$. If the elements of G are of degree 1 in the T_i , it follows that f_1, \dots, f_q is an s -sequence of M .

Let f_1, \dots, f_q be monomials of A . Set $f_{ij} = \frac{f_i}{[f_i, f_j]}$ for $i \neq j$, where $[f_i, f_j]$ is the greatest common divisor of the monomials f_i and f_j . J is generated by $g_{ij} = f_{ij}T_j - f_{ji}T_i$ for $1 \leq i < j \leq q$. The monomial sequence f_1, \dots, f_q is an s -sequence if and only if g_{ij} , for $1 \leq i < j \leq q$, is a Gröbner basis for J for any term order which extends an admissible term order on the T_i in $S = K[X_1, \dots, X_n; T_1, \dots, T_q]$. Note that the annihilator ideals of the monomial sequence f_1, \dots, f_q are the ideals $I_i = (f_{1i}, f_{2i}, \dots, f_{i-1,i})$ for $i = 1, \dots, q$.

Remark 3.2. Let I be an ideal of A generated by an s -sequence f_1, \dots, f_q of monomials with respect to some admissible term order $<$. From the theory of Gröbner bases, one has that f_1, \dots, f_q is an s -sequence with respect to any other admissible term order ([11], Lemma 1.2).

For more details on this subject see [11].

Since the property to be an s -sequence may depend on the order on the sequence, if I is a squarefree lexicographic ideal when we write

$$I = (f_1, f_2, \dots, f_q),$$

we suppose

$$f_1 >_{\text{hslex}} f_2 >_{\text{hslex}} \dots >_{\text{hslex}} f_q.$$

In order to simplify the notations we will denote $>_{\text{hslex}}$ by $>$.

For any positive integer q we set $[q] = \{1, \dots, q\}$.

The following lemma will be crucial in the sequel.

Lemma 3.3. *Let $I = (f_1, \dots, f_q) \subsetneq A$ be a squarefree lexsegment ideal generated in degree d with $G(I) \subsetneq M_d^s$.*

The following conditions are equivalent:

- (1) $[f_{ij}, f_{h\ell}] = 1$, for $i < j$, $h < \ell$, $i \neq h$, $j \neq \ell$, $i, j, h, \ell \in [q]$.
- (2) $|G(I)| \leq n - d + 2$.
- (3) I is an USLI or I is an ASLI of A .

Proof. From Corollary 2.3 and Remark 2.4, conditions (2) and (3) are equivalent, then we have only to prove that (1) \Leftrightarrow (2).

(1) \Rightarrow (2). Note that the monomial generators f_i of I are described by (2.2), for $i = 1, \dots, n - d + 2$. Suppose $|G(I)| > n - d + 2$. Since I is a squarefree lexsegment ideal of degree d , then $X_1X_2 \cdots X_{d-2}X_dX_{d+2} \in G(I)$.

From (2.2), set $t = n - d + 2$ and $t' = t + 1$, then $f_t = X_1X_2 \cdots X_{d-2}X_dX_{d+1}$ and $f_{t'} = X_1X_2 \cdots X_{d-2}X_dX_{d+2}$. Hence $f_{tt'} = X_{d+1}$. Again from (2.2), $f_{23} = X_{d+1}$. Hence $[f_{23}, f_{tt'}] = X_{d+1}$. A contradiction.

(2) \Rightarrow (1). Let $|G(I)| = q \leq n - d + 2$ and $I = (f_1, \dots, f_q)$.

If $q < n - d + 2$, then from (2.1):

$$G(I) = \{X_1X_2 \cdots X_{d-1}X_d, X_1X_2 \cdots X_{d-1}X_{d+1}, \dots, X_1X_2 \cdots X_{d-1}X_k\},$$

with $d \leq k \leq n$. Hence:

$$f_{12} = X_d, f_{13} = X_d, \dots, f_{1q} = X_d,$$

$$f_{23} = X_{d+1}, \dots, f_{2q} = X_{d+1},$$

and so on.

By the structure of $G(I)$, this computation implies that $f_{ij} \neq f_{h\ell}$ for $i \neq h$ and $j \neq \ell$. Hence $[f_{ij}, f_{h\ell}] = 1$ for $i < j$, $h < \ell$, $i \neq h$, $j \neq \ell$ and $i, j, h, \ell \in [q]$.

If $q = n - d + 2$, then from (2.2):

$$G(I) = \{X_1X_2 \cdots X_{d-1}X_d, \dots, X_1X_2 \cdots X_{d-1}X_n, X_1X_2 \cdots X_{d-2}X_dX_{d+1}\}.$$

We have:

$$\begin{aligned} f_{12} &= X_d, f_{13} = X_d, \dots, f_{1,n-d+1} = X_d, f_{1,n-d+2} = X_{d-1}, \\ f_{23} &= X_{d+1}, \dots, f_{2,n-d+1} = X_{d+1}, f_{2,n-d+2} = X_{d-1}, \dots, \\ f_{1q} &= X_{d-1}, f_{2q} = X_{d-1}, f_{3q} = X_{d-1}X_{d+2}, \dots, f_{q-1,q} = X_{d-1}X_n. \end{aligned}$$

Hence $[f_{ij}, f_{h\ell}] = 1$ for $i < j$, $h < \ell$, $i \neq h$, $j \neq \ell$ and $i, j, h, \ell \in [q]$. □

Theorem 3.4. *Let $I \subsetneq A$ be a squarefree lexsegment ideal generated in degree d with $G(I) \subsetneq M_d^s$. Then I is generated by an s -sequence if and only if $|G(I)| \leq n - d + 2$.*

Proof. Let $I = (f_1, f_2, \dots, f_q)$ be a squarefree lexsegment ideal and suppose that f_1, f_2, \dots, f_q is an s -sequence. We prove that $[f_{ij}, f_{h\ell}] = 1$, for $i < j$, $h < \ell$, $i \neq h$, $j \neq \ell$, with $i, j, h, \ell \in [q]$.

The s -sequence property implies that $G = \{g_{ij} = f_{ij}T_j - f_{ji}T_i \mid 1 \leq i < j \leq q\}$ is a Gröbner basis for J . In particular, $S(g_{ij}, g_{h\ell})$ has a standard expression with respect G with remainder 0. Note that, to get a standard expression of $S(g_{ij}, g_{h\ell})$ is equivalent to find some $g_{st} \in G$ whose initial term divides the initial term of $S(g_{ij}, g_{h\ell})$ and substitute a multiple of g_{st} such that the remaindered polynomial has a smaller initial term and so on up to the remainder is 0. We have:

$$S(g_{ij}, g_{h\ell}) = \frac{f_{ij}f_{\ell h}}{[f_{ij}, f_{h\ell}]}T_jT_h - \frac{f_{h\ell}f_{ji}}{[f_{ij}, f_{h\ell}]}T_iT_\ell.$$

First observe that $[f_{ij}, f_{j\ell}] = 1$ as f_1, \dots, f_q are squarefree monomials.

Now we consider the other cases. Suppose that $i < j$, $h < \ell$, $i \neq h$, $j \neq \ell$. As $S(g_{ij}, g_{h\ell})$ has a standard expression with respect G there exists g_{st} such that $in_{<}(g_{st})$ divides $in_{<}(S(g_{ij}, g_{h\ell}))$.

We distinguish two cases: $\ell > j$, $\ell < j$.

Let $\ell > j$, then $in_{<}(g_{st}) \mid \frac{f_{h\ell}f_{ji}}{[f_{ij}, f_{h\ell}]}$.

If $f_{h\ell} \mid \frac{f_{h\ell}f_{ji}}{[f_{ij}, f_{h\ell}]}$, then $[f_{ij}, f_{h\ell}] \mid f_{ji}$. But, since $[[f_{ij}, f_{h\ell}], f_{ji}] = 1$, it follows $[f_{ij}, f_{h\ell}] = 1$.

Consider $f_{s\ell} \mid \frac{f_{h\ell}f_{ji}}{[f_{ij}, f_{h\ell}]}$, where $f_{s\ell} = in_{<}(g_{s\ell})$ with $s < j$ and $s < h$. We can write:

$$S(g_{ij}, g_{h\ell}) = -\frac{f_{ji}f_{h\ell}}{f_{s\ell}[f_{ij}, f_{h\ell}]}g_{s\ell}T_i + \frac{f_{ij}f_{\ell h}}{[f_{ij}, f_{h\ell}]}T_jT_h - \frac{f_{ji}f_{h\ell}f_{\ell s}}{f_{s\ell}[f_{ij}, f_{h\ell}]}T_iT_s.$$

Hence $\frac{f_{ij}f_{\ell h}}{[f_{ij}, f_{h\ell}]}T_jT_h$ is divided by f_{ij} and consequently $[f_{ij}, f_{h\ell}] \mid f_{\ell h}$. But, since $[[f_{ij}, f_{h\ell}], f_{\ell h}] = 1$, then it follows $[f_{ij}, f_{h\ell}] = 1$.

Let $\ell < j$, then $in_{<}(g_{st}) \mid \frac{f_{ij}f_{\ell h}}{[f_{ij}, f_{h\ell}]}$.

If $f_{ij} \mid \frac{f_{ij}f_{\ell h}}{[f_{ij}, f_{h\ell}]}$, then $[f_{ij}, f_{h\ell}] \mid f_{\ell h}$. But, as we have $[[f_{ij}, f_{h\ell}], f_{\ell h}] = 1$, then it follows

$$[f_{ij}, f_{h\ell}] = 1.$$

Consider $f_{sh} \mid \frac{f_{ij}f_{\ell h}}{[f_{ij}, f_{h\ell}]}$, where $f_{sh} = in_{<}(g_{sh})$. We can write:

$$S(g_{ij}, g_{h\ell}) = \frac{f_{ij}f_{\ell h}}{f_{sh}[f_{ij}, f_{h\ell}]}T_jg_{sh} - \frac{f_{h\ell}f_{ji}}{[f_{ij}, f_{h\ell}]}T_iT_\ell + \frac{f_{ij}f_{\ell h}f_{hs}}{f_{sh}[f_{ij}, f_{h\ell}]}T_jT_s.$$

Hence $\frac{f_{ij}f_{\ell h}f_{hs}}{f_{sh}[f_{ij}, f_{h\ell}]}T_jT_s$ is divided by f_{ij} . Therefore $f_{sh}[f_{ij}, f_{h\ell}] \mid f_{\ell h}f_{hs}$. But as we have $[[f_{ij}, f_{h\ell}], f_{\ell h}] = 1$, then $f_{sh} \mid f_{\ell h}$ and $[f_{ij}, f_{h\ell}] \mid f_{hs}$. By the structure of the monomials f_1, \dots, f_q , if $[f_{ij}, f_{h\ell}] \mid f_{hs}$, with $s < h$, then $[f_{ij}, f_{h\ell}] = 1$.

Hence in any case we have $[f_{ij}, f_{h\ell}] = 1$ for $i < j$, $h < \ell$, $i \neq h$, $j \neq \ell$, with $i, j, h, \ell \in [q]$. It follows $|G(I)| = q \leq n - d + 2$ by Lemma 3.3.

Now suppose $|G(I)| \leq n - d + 2$. Hence from Lemma 3.3, $[f_{ij}, f_{h\ell}] = 1$, for $i < j$, $h < \ell$, $i \neq h$, $j \neq \ell$, $i, j, h, \ell \in [q]$ and the assert follows from [11] (Proposition 1.7). \square

Example 3.5. $A = K[X_1, X_2, X_3, X_4]$, $I = (X_1X_2, X_1X_3, X_1X_4, X_2X_3, X_2X_4)$.

Set $f_1 = X_1X_2$, $f_2 = X_1X_3$, $f_3 = X_1X_4$, $f_4 = X_2X_3$, $f_5 = X_2X_4$.

$G = \{f_{ij}T_j - f_{ji}T_i \mid 1 \leq i < j \leq 5\}$ is not a Gröbner basis for J . In fact J does not admit a linear Gröbner basis $BG(J)$ for any term order in $A[T_1, \dots, T_5]$: $BG(J) = \{X_2T_4 - X_3T_5, X_1T_2 - X_3T_5, X_2T_3 - X_4T_5, X_1T_1 - X_4T_5, X_3T_3 - X_4T_4, X_3T_1 - X_4T_2, X_4T_2T_3 - X_4T_1T_4\}$ [6]. Hence f_1, \dots, f_5 is not an s -sequence.

Remark 3.6. If $G(I) = M_d^s$, then $I = \mathcal{I}_d$, $2 \leq d \leq n$, where \mathcal{I}_d is the Veronese ideal of A generated by all the squarefree monomials of degree d in the variables X_1, \dots, X_n . The ideal \mathcal{I}_d is generated by an s -sequence if and only if $d = n - 1$ ([14], Theorem 2.3). Hence $I = (M_d^s)$ is generated by an s -sequence if and only if $d = n - 1$.

Proposition 3.7. *Let $I \subsetneq A$ be a squarefree lexsegment ideal generated in degree d such that $|G(I)| \leq n - d + 2$. Then the annihilator ideals of the sequence of the monomial generators of I are:*

$$I_1 = (0), \quad I_i = (X_d, \dots, X_{d+i-2}) \quad \text{for } i = 2, \dots, n - d + 1,$$

$$I_i = (X_{d-1}) \quad \text{for } i = n - d + 2.$$

Proof. Set $|G(I)| = q$. Let $I = (f_1, \dots, f_q)$ with $f_1 > \dots > f_q$.

Set $f_{ij} = \frac{f_i}{[f_i, f_j]}$ for $i < j$, $i, j \in [q]$. Then the annihilator ideals of the monomial sequence f_1, \dots, f_q are $I_i = (f_{1i}, f_{2i}, \dots, f_{i-1,i})$, for $i \in [q]$.

For $i = 1$, $I_1 = (0)$ and by the structure of these monomials, it follows:

$$I_2 = (f_{12}) = (X_d), \quad I_3 = (f_{13}, f_{23}) = (X_d, X_{d+1}), \quad \dots,$$

$$I_{n-d+1} = (f_{1,n-d+1}, f_{2,n-d+1}, \dots, f_{n-d,n-d+1}) = (X_d, X_{d+1}, \dots, X_{n-1})$$

and

$$I_{n-d+2} = (f_{1,n-d+2}, \dots, f_{n-d+1,n-d+2}) = (X_{d-1}, X_{d-1}X_{d-2}, \dots, X_{d-1}X_n) = (X_{d-1}).$$

Hence the assert follows. \square

Remark 3.8. If I is an USLI of A generated in degree d , then I is generated by a strong s -sequence. In fact $I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_{n-d+1}$ for $|G(I)| < n - d + 2$.

Theorem 3.9. Let $I \subsetneq A$ be an USLI generated in degree d . Then

- (1) $\dim(\text{Sym}_A(I)) = n + 1$;
- (2) $e(\text{Sym}_A(I)) = |G(I)|$;
- (3) $\text{reg}_A(\text{Sym}_A(I)) = 1$;
- (4) $\text{depth}_A(\text{Sym}_A(I)) = n + 1$.

Proof. (1) I is generated by a strong s -sequence. Hence, by [15] (Theorem 4.8), $\text{Sym}_A(I)$ has dimension $\dim(A) + 1 = n + 1$.

(2) Let $|G(I)| = q$. By [11] (Proposition 2.4) we have $e(\text{Sym}_A(I)) = \sum_{i=1}^q e(A/I_i)$. By Proposition 3.7, the annihilator ideals I_i are generated by a regular sequence, then, by [15] (Theorem 4.8), $e(A/I_i) = 1$, for $i = 2, \dots, q$ and $e(A/(0)) = 1$. Hence $e(\text{Sym}_A(I)) = \sum_{i=1}^q e(A/I_i) = q$.

(3) By [15] (Theorem 4.8):

$$\begin{aligned} \text{reg}(\text{Sym}_A(I)) &= \text{reg}_A(A[T_1, \dots, T_q]/J) \\ &\leq \text{reg}_A(A[T_1, \dots, T_q]/\text{in}_<(J)) = \text{reg}_A(A[T_1, \dots, T_q]/(I_1 T_1, \dots, I_q T_q)) \\ &\leq \max_{2 \leq i \leq q} \left\{ \sum_{j=1}^{i-1} \deg X_j - (i-2) \right\} = (i-1) - (i-2) = 1, \end{aligned}$$

for $i = 2, \dots, q$. Since J is generated by the linear forms of degree two $X_i T_j - X_j T_i$, for $i, j = 1, \dots, q$, then $\text{reg}_A(A[T_1, \dots, T_q]/J) \geq 1$. It follows that $\text{reg}_A(\text{Sym}_A(I)) = 1$.

(4) $\text{depth}_A(\text{Sym}_A(I)) \geq \dim(A) + 1 = n + 1$ ([15], Theorem 4.8).

Since $\text{depth}_A(\text{Sym}_A(I)) \leq \dim \text{Sym}_A(I) = n + 1$, the assert follows. \square

Remark 3.10. If I is an USLI of A generated in degree d , then $\text{Sym}_A(I)$ is Cohen-Macaulay.

Theorem 3.11. Let $I \subsetneq A$ be an AUSLI generated in degree d . Then

- (1) $\dim(\text{Sym}_A(I)) = n + 1$.
- (2) $e(\text{Sym}_A(I)) = 2|G(I)| - 2$.

Proof. (1) By [11] (Proposition 2.4), we have $\dim(\text{Sym}_A(I)) = \max\{\dim(A/(I_{i_1} + \dots + I_{i_r})) + r\}$, for $1 \leq i_1 < \dots < i_r \leq n - d + 2$.

Hence, by Proposition 3.7:

$$\dim(\text{Sym}_A(I)) = (n - r + 1) + r = n + 1.$$

(2) We have that $|G(I)| = n - d + 2$.

From [11] (Proposition 2.4),

$$e(\text{Sym}_A(I)) = \sum_{1 \leq i_1 < \dots < i_r \leq n-d+2} e(A/(I_{i_1} + \dots + I_{i_r}))$$

with $\dim(A/(I_{i_1} + \dots + I_{i_r})) = \dim(\text{Sym}_A(I)) - r = n + 1 - r$, $1 \leq r \leq n - d + 2$. By Proposition 3.7, $I_i = (X_d, \dots, X_{d+i-2})$, for $i = 2, \dots, n - d + 1$, $I_i = (X_{d-1})$ for $i = n - d + 2$.

Set $H = I_{i_1} + \dots + I_{i_r}$. Hence A/H is Cohen-Macaulay and has a linear resolution with projective dimension equal to the number of the generators of H ([10]). Then

$e(A/H) = 1$, by Huneke-Miller formula ([12]).

Set $|G(I)| = n - d + 2 = q$ and $d' = \dim(A/H) = n + 1 - r$. Then $e(\text{Sym}_A(I))$ is given by the sum of the following terms:

$e(A/(0)) = 1$, for $r = 1$ and $d' = n$,

$e(A/(I_1 + I_2)) = 1$, $e(A/(I_1 + I_q)) = 1$, for $r = 2$ and $d' = n - 1$,

$e(A/(I_1 + I_2 + I_3)) = 1$, $e(A/(I_1 + I_2 + I_q)) = 1$, for $r = 3$ and $d' = n - 2$,

$e(A/(I_1 + I_2 + I_3 + I_4)) = 1$, $e(A/(I_1 + I_2 + I_3 + I_q)) = 1$, for $r = 4$ and $d' = n - 3$,
and so on up to

$e(A/(I_1 + I_2 + \dots + I_{q-1})) = 1$, $e(A/(I_1 + \dots + I_{q-2} + I_q)) = 1$, for $r = n - d + 1$
and $d' = d - 2$,

$e(A/(I_1 + I_2 + \dots + I_q)) = 1$, for $r = n - d + 2$ and $d' = d - 1$.

Hence

$$\begin{aligned} e(\text{Sym}_A(I)) &= e(A/(0)) + 2e(A/(I_1 + I_2)) + \dots + 2e(A/(I_1 + I_2 + \dots + I_{q-1})) \\ &\quad + e(A/(I_1 + I_2 + \dots + I_q)) = 2(q - 2) + 2 = 2q - q. \end{aligned}$$

□

REFERENCES

- [1] Aramova, A., Herzog, J., Hibi, T.: Squarefree lexsegment ideals. *Math. Z.* 228, 353–378 (1998).
- [2] Babson, E., Novik, I., Thomas R.: Reverse lexicographic shifting. *J. Algebraic Combin.* 23, 107–123 (2006).
- [3] Bayer, D., Charalambous, H., Popescu, S.: Extremal Betti numbers and Applications to Monomial Ideals. *J. Algebra* 221, 497–512 (1999).
- [4] Björner, A., Kalai, G.: An extended Euler-Poincaré. *Acta Math.* 161, 279–303 (1988).
- [5] Bruns, W., Herzog, J.: Cohen-Macaulay rings. Cambridge University Press, 1996.
- [6] CoCoA team: A system for doing computations in commutative algebra. Available at <http://cocoa.dima.unige.it>.
- [7] Crupi, M., Utano, R.: Extremal Betti numbers of lexsegment ideals. In: *Lecture Notes in Pure and Applied Math., Geometric and combinatorial aspects of Commutative algebra* 217, 159–164 (2000).
- [8] Crupi, M., Utano R.: Extremal Betti numbers of graded ideals. *Results Math.* 43, 235–244 (2003).
- [9] Eisenbud, D.: *Commutative Algebra with a view towards Algebraic Geometry*. Springer-Verlag 1995.
- [10] Herzog, J., Kühl, M.: On Betti numbers of finite pure and linear resolutions. *Comm. Algebra* 12, 1627–1646 (1984).
- [11] Herzog, J., Restuccia, G., Tang, Z.: s -sequences and symmetric algebras. *Manuscripta Math.* 104, 479–501 (2001).
- [12] Huneke, C., Miller, M.: A note on the multiplicity of Cohen-Macaulay algebras with pure resolutions. *Canad. J. Math.* 37, 1149–1162 (1985).
- [13] Kalai, G.: The diameter of graphs of convex polytopes and f -vector theory. In: *Applied Geometry and Discrete Mathematics - The Victor Klee Festschrift* (Eds: P. Gritzmann and B. Sturmfels), DIMACS Series in Discrete Mathematics and Theoretical Computer Science 4, 387–411. Amer. Math. Soc., Providence, RI (1991).
- [14] La Barbiera, M., Restuccia, G.: Mixed product ideals generated by s -sequences, *Algebra Colloq.* 18(4), 553–570 (2011).
- [15] Tang, Z.: On certain monomial sequences. *J. Algebra* 282, 831–842 (2004).

UNIVERSITY OF MESSINA, DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, VIALE
FERDINANDO STAGNO D'ALCONTRES, 31, 98166 MESSINA, ITALY
E-mail address: mcrupi@unime.it

UNIVERSITY OF MESSINA, DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, VIALE
FERDINANDO STAGNO D'ALCONTRES, 31, 98166 MESSINA, ITALY
E-mail address: monicalb@unime.it